Due Sun

1.1 – Introduction to Systems of Linear Equations

A **linear equation** can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where a_i are constants, not all zero, and b is a constant.

14. In each part, use parametric equations to describe the solution set of the linear equation.

a. $x + 10y = 2$

b. $x_1 + 3x_2 - 12x_3 = 3$

c. $4x_1 + 2x_2 + 3x_3 + x_4 = 20$

d. $v + w + x - 5y + 7z = 0$

a. $x = -10y + 2$

Let $t = y$.

Then $\boxed{\begin{aligned} x &= -10t + 2 \\ y &= t \end{aligned}}$

c. $x_1 = -\frac{1}{2}x_2 - \frac{3}{4}x_3 - \frac{1}{4}x_4 + 5$

Let $r = x_2$, $s = x_3$, $t = x_4$

then

$\boxed{\begin{aligned} x_1 &= -\frac{1}{2}r - \frac{3}{4}s - \frac{1}{4}t + 5 \\ x_2 &= r \\ x_3 &= s \\ x_4 &= t \end{aligned}}$

A parameter (in this context) is a variable that when assigned a value provides a specific solution.

A **homogeneous linear equation** is a linear equation in which $b = 0$.

A **system of linear equations** is a finite set of linear equations. It has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Note that the first of the double subscripts references the equation number, and the second references the term in that equation.

The **augmented matrix** for the above linear system is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \quad \text{or} \quad \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

16. Each linear system has infinitely many solutions. Use parametric equations to describe its solution set.

a.

$$\begin{aligned} 6x_1 + 2x_2 &= -8 \\ 3x_1 + x_2 &= -4 \end{aligned}$$

Augmented Matrix:

$$\left[\begin{array}{cc|c} 6 & 2 & -8 \\ 3 & 1 & -4 \end{array}\right] \quad R_1 \to \tfrac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 3 & 1 & -4 \\ 3 & 1 & -4 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 3 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

R_1: $3x_1 + x_2 = -4 \Rightarrow x_1 = -\frac{1}{3}x_2 - \frac{4}{3}$

Let $t = x_2$.

$$\boxed{\begin{aligned} x_1 &= -\tfrac{1}{3}t - \tfrac{4}{3} \\ x_2 &= t \end{aligned}}$$

b.

$$2x - y + 2z = -4$$
$$6x - 3y + 6z = -12$$
$$-4x + 2y - 4z = 8$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & -4 \\ 2 & -1 & 2 & -4 \\ -2 & 1 & -2 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2x - y + 2z = -4$

$x = \frac{1}{2}y - z - 2$

Let $s = y$, $t = z$

then
$$\boxed{\begin{array}{l} x = \frac{1}{2}s - t - 2 \\ y = s \\ z = t \end{array}}$$

Alternative form: $(x, y, z) = (\frac{1}{2}s - t - 2, s, t)$

The following are **elementary row operations** performed on a matrix.

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

A **solution** of a linear system is a sequence of n numbers $s_1, s_2, \dots, s_n$ that when substituted for corresponding unknowns x_i makes each equation a true statement. If $n = 2$, then the solution is an **ordered pair**, and if $n = 3$, it is an **ordered triple**. In general, an **ordered n-tuple** has the form $(s_1, s_2, \dots, s_n)$.

A linear system is **consistent** if it has at least one solution.

A linear system is **inconsistent** if it has no solutions.

20. Find all values of k for which the given augmented matrix corresponds to a consistent linear system.

a. $\begin{array}{cc} x & y \end{array}$
$\left[\begin{array}{cc|c} 3 & -4 & k \\ -6 & 8 & 5 \end{array}\right]$ $3x-4y=k$

b. $\begin{array}{cc} x & y \end{array}$
$\left[\begin{array}{cc|c} k & 1 & -2 \\ 4 & -1 & 2 \end{array}\right]$

$R_2 \to R_2 + 2R_1$

$\begin{array}{ccc} 6 & -8 & 2k \\ -6 & 8 & 5 \\ \hline 0 & 0 & 2k+5 \end{array}$

$\left[\begin{array}{cc|c} 3 & -4 & k \\ 0 & 0 & 2k+5 \end{array}\right]$

R_2: $0 = 2k+5$

$k = -\frac{5}{2}$

$R_2 \to R_2 + R_1$

$\left[\begin{array}{cc|c} k & 1 & -2 \\ k+4 & 0 & 0 \end{array}\right] \to y = -2$

$(k+4)x = 0$

True if $k=-4$.

But if $k \neq -4$, then $x = 0$

All values of k work.

Ex: A 3-7-9 diet calls for 3 units of fat, 7 units of protein, and 9 units of carbs in each meal. Suppose an individual has three possible foods to choose from to meet these requirements. Each ounce of the food contains

x Food 1: 3 units of fat, 4 units of protein, and 1 unit of carbs
y Food 2: 2 units of fat, 5 units of protein, 3 units of carbs
z Food 3: 4 units of fat, 1 unit of protein, 2 units of carbs

Let x, y, and z denote the number of ounces of the first, second, and third foods that a person will consume at the main meal. Find a linear system in x, y, and z whose solution tells how many ounces of each food must be consumed to meet the diet requirements.

Fat: $3x + 2y + 4z = 3$

Protein: $4x + 5y + z = 7$

Carbs: $x + 3y + 2z = 9$